University of Tikrit

College of Engineering

Electrical Engineering Department

Electromagnetic Fields

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Class : Second	**Theory :** 2 Hrs. /Wk
Subject: Electromagnetic Fields	**Tutorial :** 1 Hrs. /Wk
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Article
Vectors Analysis: *Vector components, Unit vector, Vector algebra, Rectangular Coordinate , Cylindrical and Spherical Coordinates*
Coulombs Law and Electric Field Intensity: *Coulombs Law, Electric Field Intensity, Field of n Point Charge, Field Due to a Continuous Volume Charge Distribution, Field of a Line Charge and a sheet charge , Streamlines and Sketches of Fields*
Electric Flux Density, Gauss' Law, and Divergence: *Electric Flux Density, Gauss's Law, Applications of Gauss' Law, Divergence, Maxwell's First Equation*
Energy and Potential: *Energy and Potential in a Moving Point Charge in an Electric Field, Definition of Potential Difference and Potential, Potential Field of a Point Charge, Potential Field of a System of Charges, Potential Gradient, Dipole*
Conductor , Dielectric and Capacitance: *Current and Current Density, Continuity of Current, Metallic Conductors, Conductor Properties and Boundary Conditions, The Nature of Dielectric Materials, Boundary Conditions for Perfect Dielectric Materials, Capacitance, Poisson's and Laplace's Equations, Examples of the Solution of Laplace's Equation, Example of the Solution of Poisson's Equation*
The Steady Magnetic Field: *Biot-Savart Law, Ampere's Circuital Law, Stokes' Theorem, Magnetic Flux and Magnetic Flux Density, Inductance, Scalar and Vector Magnetic Potentials*
Magnetic Forces, Materials and Inductance: *Force on a Moving Charge, Force on a Differential Current Element, Force Between Differential Current Elements, Force and Torque on a Closed Circuit, Magnetization and Permeability, Magnetic Boundary Conditions, The Magnetic Circuit*
Time-Varying Fields and Maxwell's Equations: *Faraday's Law, Induced voltage, Displacement Current, Maxwell's Equations in Point Form,*
Uniform Plane Wave: *Wave Equation, Wave Propagation in Free Space, Wave Propagation in Dielectrics, Wave Propagation in Good Conductors*

Textbook: 1- **Engineering Electromagnetics** , William H. Hayt, Published By Mcgraw-Hill,

2- **Elements of Electromagnetics** , Matthew N.O. Sadiku

Useful Web pages:

1- http://www.youtube.com/watch?v=686R2TWCIcI&list=PLEJFWUfvvpOFRKj7inlmgJrL1YW9j089r
2- https://www.facebook.com/groups/265743656900260/

1. Scalars and Vector

The term ***scalar*** refers to a quantity whose value may be represented by a single (positive or negative) real number e.g. length, time, voltage etc. A scalar is represented simply by a letter e.g. *A*, *B* and *E*, or by $|\mathrm{A}|$, $|\mathrm{B}|$ and $|\mathrm{E}|$.

A ***vector*** quantity has both a magnitude and a direction in space. Force, velocity and electric field are examples of vectors. Each quantity is characterized by both a magnitude and a direction. A vector is represented by a letter with an arrow on top of it, such as $\vec{\mathrm{A}}$, $\vec{\mathrm{B}}$ and $\vec{\mathrm{E}}$, or by a letter in boldface type such as **A**, **B** and **E**

1.1 Vector Addition and Subtraction

Two vectors **A** and **B** can be added together to give another vector **C**: i.e.

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

The vector addition is carried out component by component. Thus,

if $\vec{\mathbf{A}} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$ and $\vec{\mathbf{B}} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$, then

$$\vec{\mathrm{A}} + \vec{\mathrm{B}} = (A_x + B_x)\mathbf{a}_\mathbf{x} + \left(A_y + B_y\right)\mathbf{a}_\mathbf{y} + (A_z + B_z)\mathbf{a}_\mathbf{z}$$

Vector subtraction is similarly carried out as

$$\therefore \vec{\mathrm{A}} - \vec{\mathrm{B}} = (A_x - B_x)\mathbf{a}_x + \left(A_y - B_y\right)\mathbf{a}_y + (A_z - B_z)\mathbf{a}_z$$

Example: If $\vec{\mathbf{A}} = 3\mathbf{a}_x + 7\mathbf{a}_y + 2\mathbf{a}_z$ and $\vec{\mathbf{B}} = 5\mathbf{a}_y + 8\mathbf{a}_z$. Find **A**+**B** and **A**-**B**?

Solution:

$A_x = 3,\qquad A_y = 7,\qquad A_z = 2$

$B_x = 0,\qquad B_y = 5,\qquad B_z = 8$

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = (3+0)\mathbf{a}_\mathbf{x} + (7+5)\mathbf{a}_\mathbf{y} + (2+8)\mathbf{a}_\mathbf{z} \quad = 3\mathbf{a}_\mathbf{x} + 12\mathbf{a}_\mathbf{y} + 10\mathbf{a}_\mathbf{z}$$

$$\vec{\mathbf{A}} - \vec{\mathbf{B}} = (3-0)\mathbf{a}_\mathbf{x} + (7-5)\mathbf{a}_\mathbf{y} + (2-8)\mathbf{a}_\mathbf{z} \quad = 3\mathbf{a}_\mathbf{x} + 2\mathbf{a}_\mathbf{y} - 6\mathbf{a}_\mathbf{z}$$

2. The Cartesian Coordinate System

In the Cartesian coordinate system we set up three coordinate axes mutually at right angles to each other, and call them the x, *y,* and z axes. It is customary to choose a *right-handed* coordinate system, in which a rotation (through the smaller angle) of the *x* axis into the *y* axis would cause a right-handed screw to progress in the direction of the *z* axis. Figure 1.3 shows a right-handed cartesian coordinate system.

A point is located by giving its x, y, and z coordinates. These are, respectively, the distances from the origin to the intersection of a perpendicular dropped from the point to the x, y, and z axes.

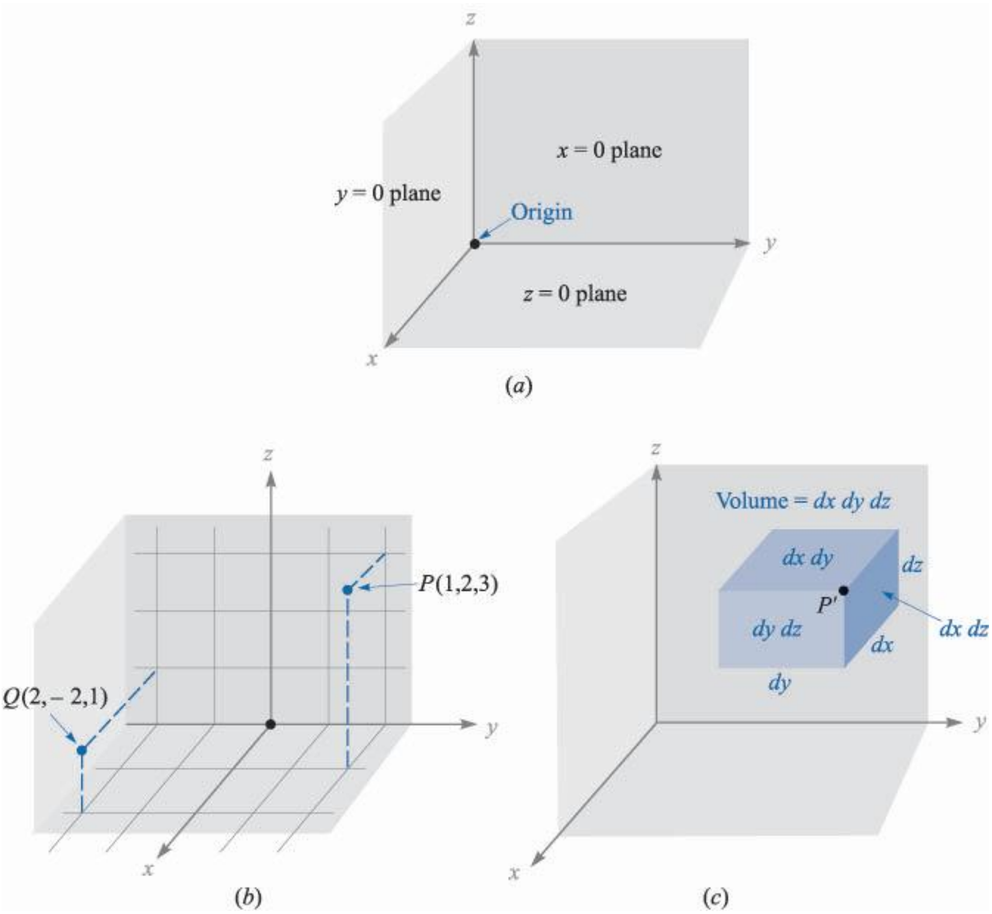


Figure 1.3 (a) A right-handed cartesian coordinate system. (b) The location of points *P(1,* 2, 3) and *Q(2,* —2, 1). (c) The differential volume element in Cartesian coordinates

Also shown in Figure 1.3(c) are differential element in length, area, and volume. Notes from the figure that in Cartesian coordinate:

1- Differential displacement is given by

$$dL = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$$

2- Differential normal area is given by

$$dS = dydz\,\mathbf{a}_x$$

$$dS = dxdz\,\mathbf{a}_y$$

$$dS = dxdy\,\mathbf{a}_z$$

3- Differential volume is given by

$$dV = dxdydz$$

3. Vector Components and Unit Vectors

A vector $\vec{\boldsymbol{A}}$ in Cartesian coordinates may be represented as $\vec{\mathbf{A}} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$. Where A_x, A_y and A_z are called the components of $\vec{\boldsymbol{A}}$ in x, y, and z directions respectively. A vector $\vec{\boldsymbol{A}}$ has both magnitude and direction. The magnitude of $\vec{\boldsymbol{A}}$ is scalar written as A or $|\vec{\boldsymbol{A}}|$.

$$|\vec{\boldsymbol{A}}| = \sqrt{\boldsymbol{A}_x^2 + \boldsymbol{A}_y^2 + \boldsymbol{A}_z^2}$$

A unit vector $\mathbf{a}_A$ along **A** is defined as a vector whose magnitude is unity (i.e., 1) and its direction is along $\vec{\boldsymbol{A}}$, i.e.

$$\mathrm{a}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{A_x\mathrm{a}_x + A_y\mathrm{a}_y + A_z\mathrm{a}_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

Notes that $|\mathbf{a}_A| = 1$

The vector between two points P(x_0, y_0, z_0) and Q(x_1, y_1, z_1) is given by:

$$\overrightarrow{PQ} = (x_1 - x_0)\mathrm{a}_x + (y_1 - y_0)\mathrm{a}_y + (z_1 - z_0)\mathrm{a}_z$$

Example: Find the vector between the points P (1, 4, 2) and Q (3, 1, 6)?

Solution: $\overrightarrow{\boldsymbol{PQ}} = (3-1)\mathbf{a}_x + (1-4)\mathbf{a}_y + (6-2)\mathbf{a}_z \quad = 2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z$

$$\overrightarrow{\boldsymbol{QP}} = (1-3)\mathbf{a}_x + (4-1)\mathbf{a}_y + (2-6)\mathbf{a}_z \quad = -2\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z$$

Example: If $\vec{\mathbf{A}} = 10\mathbf{a}_\mathrm{x} - 4\mathbf{a}_\mathrm{y} + 6\mathbf{a}_\mathrm{z}$ and $\vec{\mathbf{B}} = 2\mathbf{a}_\mathrm{x} + \mathbf{a}_\mathrm{y}$ **,** find (a) the component of **A** along $\mathbf{a}_\mathrm{y}$, (b) the magnitude of **A**, (c) the magnitude of **B**, (d) the magnitude of 3**A**-**B**, (e) the unit vector of **A**

Solution:

(a) $\;the\ component\ of\ A\ along\ a_y\ is\ A_y = -4a_y$

(b) $\;\overrightarrow{|\mathrm{A}|} = \sqrt{10^2 + 4^2 + 6^2} = 12.32$

(c) $\;\overrightarrow{|\mathrm{B}|} = \sqrt{2^2 + 1^2} = 2.23$

(d) $\;the\ magnitude\ of\ 3\boldsymbol{A} - \boldsymbol{B}$

$$3\mathbf{A} - \mathbf{B} = \mathbf{3}(10\mathbf{a}_\mathrm{x} - 4\mathbf{a}_\mathrm{y} + 6\mathbf{a}_\mathrm{z}) - (2\mathbf{a}_\mathrm{x} + \mathbf{a}_\mathrm{y}) = 28\mathbf{a}_\mathrm{x} - 5\mathbf{a}_\mathrm{y} + 6\mathbf{a}_\mathrm{z}$$

$$|3\mathbf{A} - \mathbf{B}| = \sqrt{28^2 + 5^2 + 6^2} = 29.06$$

(e) $The\ unit\ vector\ of\ A\ is$:

$$\mathrm{a}_A = \frac{A}{|A|} = \frac{10\mathbf{a}_\mathrm{x} - 4\mathbf{a}_\mathrm{y} + 6\mathbf{a}_\mathrm{z}}{12.32}$$

4. <u>The Dot Product</u>

The dot product of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, written as $\vec{\mathbf{A}}\,.\,\vec{\mathbf{B}}$, is defined geometrically as the dot product of the magnitude of $\vec{\mathbf{B}}$ and the projection of $\vec{\mathbf{A}}$ onto $\vec{\mathbf{B}}$ (or vice versa).

Thus:

$$\vec{\mathrm{A}}\,.\,\vec{\mathrm{B}} = |\vec{\mathrm{A}}||\vec{\mathrm{B}}| \cos\theta_{AB}$$

where $\boldsymbol{\theta_{AB}}$ is the smaller angle between **A** and **B**. the result of **A.B** is called either *scalar product* since it is scalar, or *dot product* due to sign.

If $\vec{\mathbf{A}} = A_x\mathbf{a_x} + A_y\mathbf{a_y} + A_z\mathbf{a_z}$ and $\vec{\mathbf{B}} = B_x\mathbf{a_x} + B_y\mathbf{a_y} + B_z\mathbf{a_z}$, then

$$\vec{\mathbf{A}}\,.\vec{\mathbf{B}} = A_x\mathrm{B}_x + A_y\mathrm{B}_y + A_z\mathrm{B}_z$$

note that

$$\mathrm{a}_x.\mathrm{a}_y = \mathrm{a}_y.\mathrm{a}_z = \mathrm{a}_z.\mathrm{a}_x = 0 \qquad (\theta = 90\,, \quad \cos\theta = \cos 90 = 0)$$

$$\mathrm{a}_x.\mathrm{a}_x = \mathrm{a}_y.\mathrm{a}_y = \mathrm{a}_z.\mathrm{a}_z = 1 \qquad (\theta = 0\,, \quad \cos\theta = \cos 0 = 1)$$

Example: The three vertices of a triangle are located at *A*(6, -1, 2), *B*(-2, 3, -4) and *C*(-3, 1, 5). Find: (a) $\vec{\boldsymbol{R}}_{\boldsymbol{AB}}$; (b) $\vec{\boldsymbol{R}}_{\boldsymbol{AC}}$; (c) the angle θ_{BAC} at vertex *A* ?

Solution:

$$\vec{R}_{AB} = (-2-6)\mathrm{a}_x + (3-(-1))\mathrm{a}_y + (-4-2)\mathrm{a}_z$$

$$\vec{R}_{AB} = -8\mathrm{a}_x + 4\mathrm{a}_y - 6\mathrm{a}_z$$

$$\vec{R}_{AC} = (-3-6)\mathrm{a}_x + (1-(-1))\mathrm{a}_y + (5-2)\mathrm{a}_z$$

$$\vec{R}_{AC} = -9\mathrm{a}_x + 2\mathrm{a}_y + 3\mathrm{a}_z$$

$$\left|\vec{R}_{AB}\right| = \sqrt{8^2 + 4^2 + 6^2} = 10.77$$

$$\left|\vec{R}_{AC}\right| = \sqrt{9^2 + 2^2 + 3^2} = 9.69$$

$$\vec{R}_{AB}\,.\vec{R}_{AC} = A_x\mathrm{B}_x + A_y\mathrm{B}_y + A_z\mathrm{B}_z = -8(-9) + 4(2) - 6(3) = 62$$

$$\vec{R}_{AB}\,.\vec{R}_{AC} = \left|\vec{R}_{AB}\right|\left|\vec{R}_{AC}\right|\cos\theta_{BAC}$$

$$62 = 10.77(9.69)\cos\theta_{BAC}$$

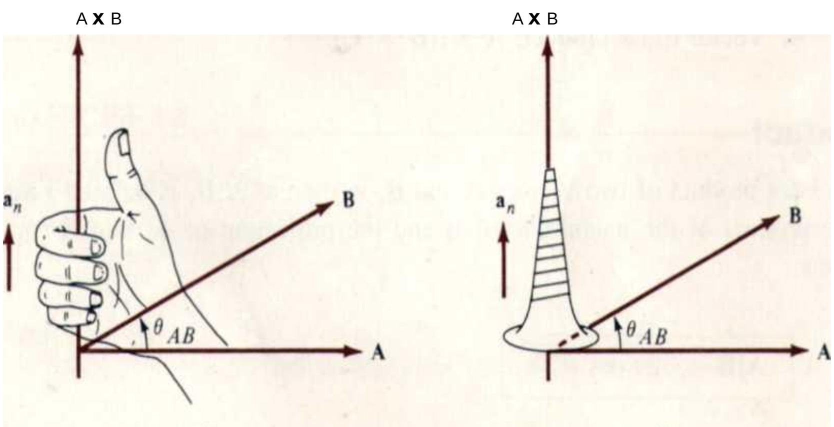
$$\theta_{BAC} = \cos^{-1}\frac{62}{104.36} = 53.55^0$$

5. The Cross Product

The cross product of two vectors A and B, written as A x B, is defined as

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta_{AB} \; a_N$$

Where $\mathbf{a}_N$ is a unit vector normal to the plane containing $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$**.** The direction of $\mathbf{a}_N$ is taken as the direction of the right thumb when the fingers of the right hand rotate from **A** to **B** as shown in Fig. 1.5(a). Alternatively, the direction of $\mathbf{a}_N$ is taken as that of the advance of a right-handed screw as **A** is turned into **B** as shown in Fig. 1.5(b).



(a) (b)

Figure 1.5: Direction of $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ using (a) right hand rule (b) a right-handed screw

The vector multiplication of equation above is called *cross product* due to the cross sign; it is also called *vector product* since the result is a vector.

If $\vec{\mathbf{A}} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$ and

$\vec{\mathbf{B}} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$, then

$$\vec{A} \times \vec{B} = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_yB_z - B_yA_z)\mathbf{a_x} + (A_zB_x - B_zA_x)\mathbf{a_y} + (A_xB_y - B_x A_y)\mathbf{a_z}$$

Note that

$\mathrm{a}_x \times \mathrm{a}_x = \mathrm{a}_y \times \mathrm{a}_y = \mathrm{a}_z \times \mathrm{a}_z = 0$ $\quad (\theta = 0, \quad \sin\theta = \sin 0 = 0)$

$\mathrm{a}_x \times \mathrm{a}_y = \mathrm{a}_z$

$\mathrm{a}_y \times \mathrm{a}_z = \mathrm{a}_x$

$\mathrm{a}_z \times \mathrm{a}_x = \mathrm{a}_y$

While

$\mathrm{a}_y \times \mathrm{a}_x = -\mathrm{a}_z$

Example: If three points *A*(6, -1, 2), *B*(-2, 3, -4) and *C*(-3, 1, 5). Find $\vec{R}_{AB} \times \vec{R}_{AC}$

Solution:

$$\vec{R}_{AB} = (-2-6)\mathrm{a}_x + (3-(-1))\mathrm{a}_y + (-4-2)\mathrm{a}_z$$

$$\vec{R}_{AB} = -8\mathrm{a}_x + 4\mathrm{a}_y - 6\mathrm{a}_z$$

$$\vec{R}_{AC} = (-3-6)\mathrm{a}_x + (1-(-1))\mathrm{a}_y + (5-2)\mathrm{a}_z$$

$$\vec{R}_{AC} = -9\mathrm{a}_x + 2\mathrm{a}_y + 3\mathrm{a}_z$$

$$|\vec{R}_{AB}| = \sqrt{8^2 + 4^2 + 6^2} = 10.77$$

$$|\vec{R}_{AC}| = \sqrt{9^2 + 2^2 + 3^2} = 9.69$$

$$\vec{R}_{AB} \times \vec{R}_{AC} = \begin{vmatrix} \mathrm{a}_x & \mathrm{a}_y & \mathrm{a}_z \\ R_{AB\,x} & R_{AB\,y} & R_{AB\,z} \\ R_{AC\,x} & R_{AC\,y} & R_{AC\,z} \end{vmatrix} = \begin{vmatrix} \mathrm{a}_x & \mathrm{a}_y & \mathrm{a}_z \\ -8 & 4 & -6 \\ -9 & 2 & 3 \end{vmatrix}$$

$$\vec{R}_{AB} \times \vec{R}_{AC} = (4*3 - 2*-6)\mathbf{a_x} + (-6*-9 - 3*-8)\mathbf{a_y} + (-8*2 - (-9)*4)\mathbf{a_z}$$

$$\vec{R}_{AB} \times \vec{R}_{AC} = 24\mathbf{a_x} + 78\mathbf{a_y} + 20\mathbf{a_z}$$

$$|\vec{R}_{AB} \times \vec{R}_{AC}| = \sqrt{24^2 + 78^2 + 20^2} = |\vec{R}_{AB}||\vec{R}_{AC}|\sin\theta_{BAC} \qquad , \quad \therefore\ \theta_{BAC} = 53.6^0$$

6. The Cylindrical Coordinate System

The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry. A vector $\vec{\mathbf{A}}$ in cylindrical coordinates can be written as:

$$\vec{\mathbf{A}} = A_\rho \mathbf{a}_\rho + A_\emptyset \mathbf{a}_\emptyset + A_z \mathbf{a}_z$$

Where $\mathbf{a}_\rho$, $\mathbf{a}_\emptyset$ and $\mathbf{a}_z$ are unit vectors in the $\rho, \emptyset$, and z-directions as illustrated in Figure 1.6.

The magnitude of $\vec{\mathbf{A}}$ is:

$$|\vec{\mathbf{A}}| = \sqrt{A_\rho^2 + A_\emptyset^2 + A_z^2}$$

A point P in cylindrical coordinates is represented as $(\rho, \emptyset, z)$ and is as shown in Figure 1.6. Observe Figure 1.6 closely and note how we define each space variable: ρ is the radius of the cylinder passing through P or the radial distance from the z-axis; $\emptyset$ is (called the *azimuthal* angle) measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system.

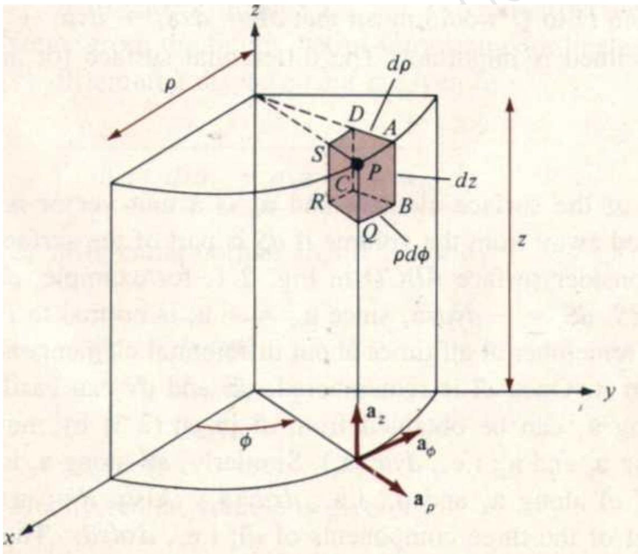


Figure 1.6: Point P, unit vector, and differential element in cylindrical coordinates

Notice that the unit vectors $\mathbf{a}_\rho$, $\mathbf{a}_\emptyset$ and $\mathbf{a}_z$ are mutually perpendicular since our coordinate system is orthogonal; $\mathbf{a}_\rho$ points in the direction of increasing $\boldsymbol{\rho}$, $\mathbf{a}_\emptyset$, in the direction of increasing $\emptyset$, and $\mathbf{a}_z$ in the positive z-direction. Thus

$$\mathrm{a}_\rho . \mathrm{a}_\rho = \mathrm{a}_\emptyset . \mathrm{a}_\emptyset = \mathrm{a}_z . \mathrm{a}_z = 1$$

$$a_\rho . a_\emptyset = a_\emptyset . a_z = a_z . a_\rho = 0$$

$$a_\rho \times a_\emptyset = a_z$$

$$a_\emptyset \times a_z = a_\rho$$

$$a_z \times a_\rho = a_\emptyset$$

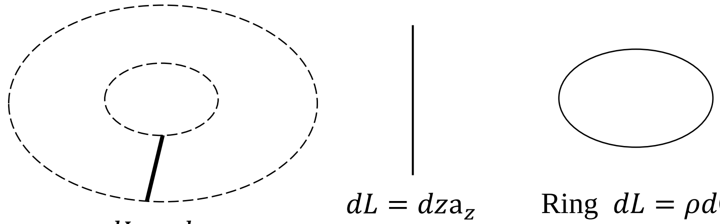
Note Also from Figure 1.6 that in cylindrical coordinate, differential element can be found:

(i) Differential displacement is given by:

$$dL = d\rho \mathbf{a_\rho}$$

$$dL = \rho d\emptyset \mathbf{a_\emptyset}$$

$$dL = dz \mathbf{a_z}$$



or $dL = d\rho \mathbf{a_\rho} + \rho d\emptyset \mathbf{a_\emptyset} + dz \mathbf{a_z}$

(ii) Differential normal area is given by:

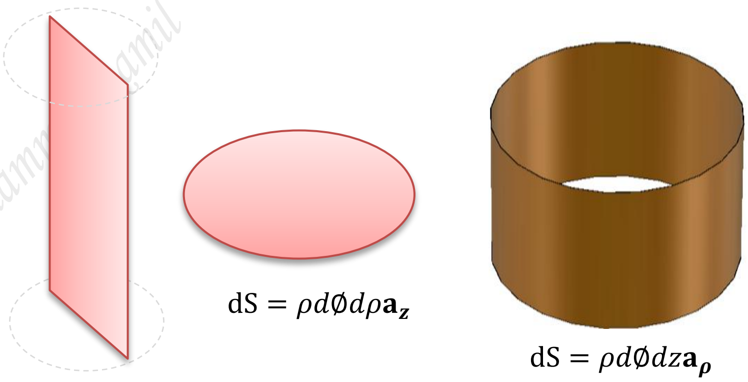
$$dS = \rho d\emptyset dz \mathbf{a_\rho}$$

$$dS = d\rho d_z \mathbf{a_\emptyset}$$

$$dS = \rho d\emptyset d\rho \mathbf{a_z}$$

(iii) Differential volume is given by:

$$dV = \rho d\rho d\emptyset dz$$



The distance between two points in cylindrical coordinate $P_1(\rho_1, \emptyset_1, z_1)$ and $P_2(\rho_2, \emptyset_1, z_1)$ is given by

$$d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\emptyset_2 - \emptyset_1) + (z_2 - z_1)^2}$$

- ***Cartesian** to **Cylindrical** Coordinate Transformation*

The relationships between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system $(\rho, \emptyset, z)$ are easily obtained as

$$\rho = \sqrt{x^2 + y^2} \ , \qquad \emptyset = \tan^{-1}\frac{y}{x} \ , \qquad z = z$$

In matrix form, we have transformation of vector $\vec{\mathbf{A}}$

from Cartesian coordinate $\vec{\mathbf{A}} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$

to cylindrical coordinate $\vec{\mathbf{A}} = A_\rho\mathbf{a}_\rho + A_\emptyset\mathbf{a}_\emptyset + A_z\mathbf{a}_z$ as

$$\begin{vmatrix} A_\rho \\ A_\emptyset \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix}$$

- ***Cylindrical** to **Cartesian** Coordinate Transformation*

The relationships between the variables $(\rho, \emptyset, z)$ of the cylindrical coordinate system and those of the Cartesian system (x, y, z) are easily obtained as

$$x = \rho\ \cos\emptyset \ , \qquad y = \rho\ \sin\emptyset \qquad z = z$$

In matrix form, we have transformation of vector $\vec{\mathbf{A}}$

From cylindrical coordinate $\vec{\mathbf{A}} = A_\rho\mathbf{a}_\rho + A_\emptyset\mathbf{a}_\emptyset + A_z\mathbf{a}_z$

To Cartesian coordinate $\vec{\mathbf{A}} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$ as

$$\begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{-y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_\rho \\ A_\emptyset \\ A_z \end{vmatrix}$$

Example: Given point P(-2, 6, 3) and vector $\overrightarrow{\mathbf{A}} = y\mathbf{a}_x + (x+z)\mathbf{a}_y$, express P and $\overrightarrow{\mathbf{A}}$ in cylindrical coordinate. Evaluate $\overrightarrow{\mathbf{A}}$ at P in Cartesian and cylindrical system?

Solution:

The vector A in Cartesian coordinate at P is:

$$\overrightarrow{\mathrm{A}} = 6\mathrm{a}_x + (-2+3)\mathrm{a}_y = 6\mathrm{a}_x + \mathrm{a}_y$$

$$|A| = \sqrt{6^2 + 1^2} = 6.08$$

The point P in cylindrical coordinate is:

$$\rho = \sqrt{x^2 + y^2} \;\; = \sqrt{-2^2 + 6^2} = 6.324$$

$$\emptyset = \tan^{-1}\frac{y}{x} \;\; = \tan^{-1}\frac{6}{-2} = 108.43$$

$$z = z = 3$$

P(6.324 , 108.43^0, 3)

The vector A in cylindrical coordinate is:

$$\begin{vmatrix} A_\rho \\ A_\emptyset \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} y \\ x+z \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} A_\rho \\ A_\emptyset \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \rho\;\sin\emptyset \\ \rho\;\cos\emptyset + z \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} A_\rho \\ A_\emptyset \\ A_z \end{vmatrix} = \begin{vmatrix} \rho\;\sin\emptyset\cos\emptyset + \rho\cos\emptyset\sin\emptyset + z\sin\emptyset \\ -\rho\sin^2\emptyset + \rho\cos^2\emptyset + z\cos\emptyset \\ 0 \end{vmatrix}$$

$$A_\rho = \rho\;\sin\emptyset\cos\emptyset + \rho\cos\emptyset\sin\emptyset + z\sin\emptyset$$

$$A_\emptyset = -\rho\sin^2\emptyset + \rho\cos^2\emptyset + z\cos\emptyset$$

$$A_z = 0$$

$A = A_\rho \mathbf{a}_\rho + A_\emptyset \mathbf{a}_\emptyset$

$A \text{ at point } P \text{ is}$:

$A_\rho = 6.324 \ \sin 108.43 \cos 108.43 + 6.324 \ \cos 108.43 \sin 108.43 + 3 \sin 108.43 = -0.948$

$A_\emptyset = -6.324 \ \sin^2 108.43 + 6.324 \cos^2 108.43 + 3 \cos 108.43 = -6.008$

$A = -0.948\mathbf{a}_\rho - 6.008\mathbf{a}_\emptyset$

$|A| = \sqrt{0.948^2 + 6.008^2} = 6.08$

7. <u>The Spherical Coordinate System</u>

The Spherical coordinate system is most appropriate when dealing with problems having spherical symmetry. A vector $\vec{\mathbf{A}}$ in spherical coordinates can be written as:

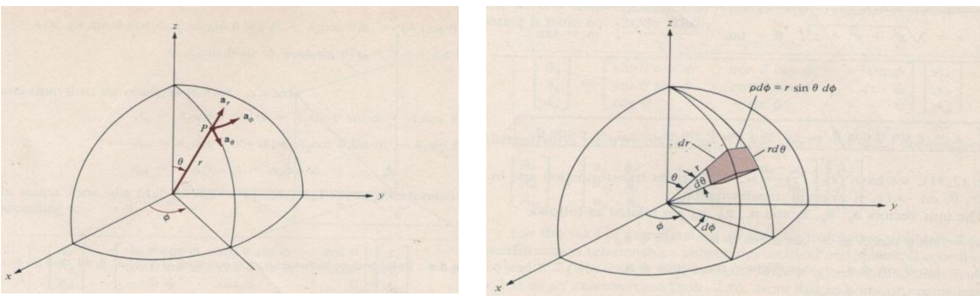
$$\vec{\mathbf{A}} = A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta + A_\emptyset \mathbf{a}_\emptyset$$

Where $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\emptyset$ are unit vectors in the r, θ , and $\emptyset$ directions.

The magnitude of $\vec{\mathbf{A}}$ is:

$$|\vec{\mathbf{A}}| = \sqrt{A_r^2 + A_\theta^2 + A_\emptyset^2}$$

A point P in spherical coordinates is represented as $(r, \theta, \emptyset)$ and is illustrate in Figure 1.7(a). From this Figure, we notice that r is defined as the distance from the origin to the point P or the radius of a sphere centred at the origin and passing through P; θ is the angle between the z-axis and the position vector of P; and $\emptyset$ is measured from the x-axis



(a) (b)

Figure 1.7: Spherical coordinate system (a) point P and unit vector (b)

Notice that the unit vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\emptyset$ are mutually perpendicular since our coordinate system is orthogonal; $\mathbf{a}_r$ points in the direction of increasing r, $\mathbf{a}_\theta$, in the direction of increasing $\boldsymbol{\theta}$, and $\mathbf{a}_\emptyset$ in the direction of increasing $\emptyset$. Thus

$$\mathrm{a}_r.\mathrm{a}_r = \mathrm{a}_\theta.\mathrm{a}_\theta = \mathrm{a}_\emptyset.\mathrm{a}_\emptyset = 1$$

$$\mathrm{a}_r.\mathrm{a}_\theta = \mathrm{a}_\theta.\mathrm{a}_\emptyset = \mathrm{a}_\emptyset.\mathrm{a}_r = 0$$

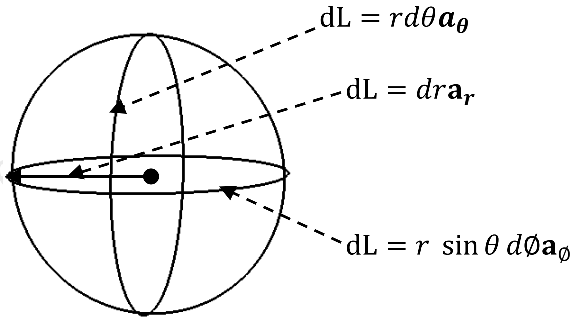
$$\mathrm{a}_r \times \mathrm{a}_\theta = \mathrm{a}_\emptyset$$

$$\mathrm{a}_\theta \times \mathrm{a}_\emptyset = \mathrm{a}_r$$

$$\mathrm{a}_\emptyset \times \mathrm{a}_r = \mathrm{a}_\theta$$

From Figure 1.7(b), we note that in spherical coordinate, differential element can be found:

(iv) Differential displacement is given by:



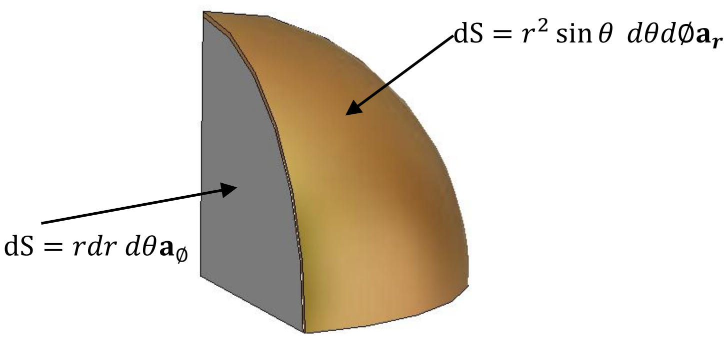
$$\mathrm{dL} = dr\mathbf{a}_{\boldsymbol{r}}$$

$$\mathrm{dL} = rd\theta\mathbf{a}_{\boldsymbol{\theta}}$$

$$\mathrm{dL} = r\ \sin\theta\ d\emptyset\mathbf{a}_\emptyset$$

or $\mathrm{dL} = dr\mathbf{a}_{\boldsymbol{r}} + rd\theta\mathbf{a}_{\boldsymbol{\theta}} + r\ \sin\theta\ d\emptyset\ \mathbf{a}_\emptyset$

(v) Differential normal area is given by:



$$\mathrm{dS} = r^2\sin\theta\ \ d\theta d\emptyset\mathbf{a}_{\boldsymbol{r}}$$

$$\mathrm{dS} = r\sin\theta\ drd\emptyset\mathbf{a}_{\boldsymbol{\theta}}$$

$$\mathrm{dS} = rdr\ d\theta\mathbf{a}_\emptyset$$

(vi) Differential volume is given by:

$$\mathrm{dV} = r^2\sin\theta\ drd\theta d\emptyset$$

The distance between two points in spherical coordinate $\mathrm{P}_1(r_1, \theta_1, \emptyset_1)$ and $\mathrm{P}_2(r_2, \theta_2, \emptyset_2)$ is given by

$$\mathrm{d} = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta_2\cos\theta_1 - 2r_1r_2\sin\theta_2\sin\theta_1\cos(\emptyset_2 - \emptyset_1)}$$

- ***Cartesian** to **Spherical** Coordinate Transformation*

The relationships between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system $(r, \theta, \emptyset)$ are easily obtained as

$$r = \sqrt{x^2 + y^2 + z^2}\ , \qquad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\ , \qquad \emptyset = \tan^{-1}\frac{y}{x}\ ,$$

In matrix form, we have transformation of vector $\vec{\mathbf{A}}$ from Cartesian coordinate $\vec{\mathbf{A}} = A_x\mathbf{a_x} + A_y\mathbf{a_y} + A_z\mathbf{a_z}$ to spherical coordinate $\vec{\mathbf{A}} = A_r\mathbf{a_r} + A_\theta\mathbf{a_\theta} + A_\emptyset\mathbf{a_\emptyset}$ as

$$\begin{vmatrix} A_r \\ A_\theta \\ A_\emptyset \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\emptyset & \sin\theta\sin\emptyset & \cos\theta \\ \cos\theta\cos\emptyset & \cos\theta\sin\emptyset & -\sin\theta \\ -\sin\emptyset & \cos\emptyset & 0 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix}$$

Figure 1.8 shows the relation between space variables

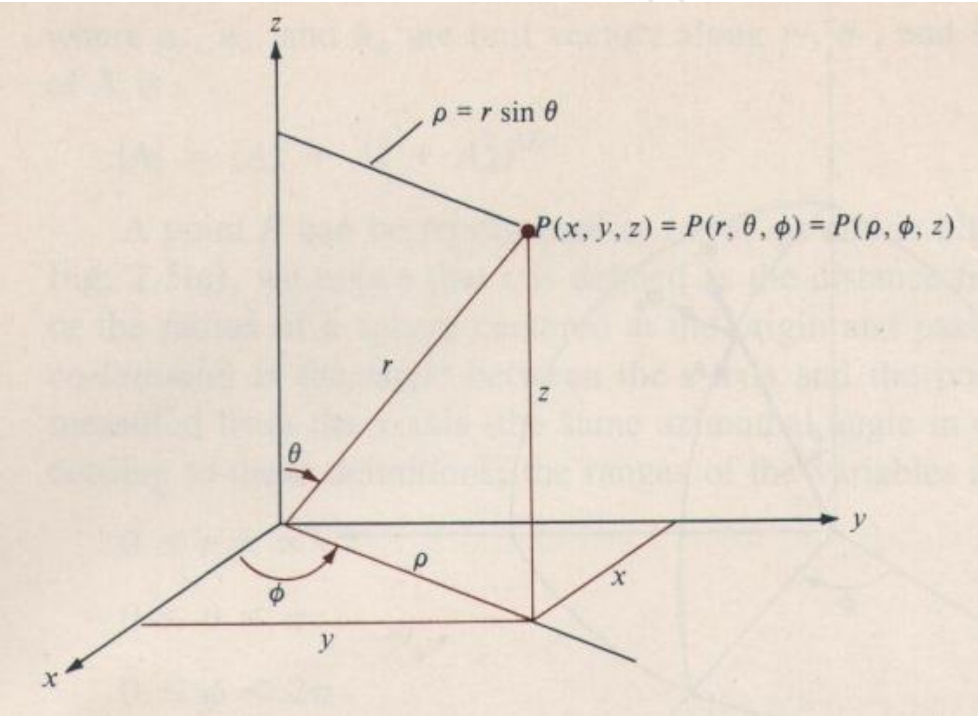


Figure 1.8: The relation between space variables (x, y, z), $(r, \theta, \emptyset)$ and $(r, \emptyset, z)$

- ***Spherical** to **Cartesian** Coordinate Transformation*

The relationships between the variables $(r, \theta, \emptyset)$ of the spherical coordinate system and those of the Cartesian system (x, y, z) are easily obtained as

$$x = r \ \sin\theta \cos\emptyset \ , \qquad y = r \ \sin\theta \sin\emptyset \qquad z = \mathrm{r}\cos\theta$$

In matrix form, we have transformation of vector $\vec{\mathbf{A}}$ from spherical coordinate $\vec{\mathbf{A}} = A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\emptyset\mathbf{a}_\emptyset$ to Cartesian coordinate $\vec{\mathbf{A}} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$ as

$$\begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{x\,z}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} & \dfrac{-y}{\sqrt{x^2+y^2}} \\ \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{y\,z}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} & \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{z}{\sqrt{x^2+y^2+z^2}} & \dfrac{-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{vmatrix} \begin{vmatrix} A_r \\ A_\theta \\ A_\emptyset \end{vmatrix}$$

Example: Express $\vec{\mathbf{E}} = \frac{\mathbf{10}}{\boldsymbol{r}}\mathbf{a}_r + r\ \cos\theta\,\mathbf{a}_\theta + \mathbf{a}_\emptyset$ in Cartesian coordinate? Find $\vec{\mathbf{E}}$ at (-3, 4, 0)?

Solution:

$$\frac{10}{r} = \frac{10}{\sqrt{x^2+y^2+z^2}}\ , \qquad r\ \cos\theta = z\ , \qquad 1 = 1$$

$$\begin{vmatrix} E_x \\ E_y \\ E_z \end{vmatrix} = \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{x\,z}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} & \dfrac{-y}{\sqrt{x^2+y^2}} \\ \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{y\,z}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} & \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{z}{\sqrt{x^2+y^2+z^2}} & \dfrac{-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{vmatrix} \begin{vmatrix} \dfrac{10}{\sqrt{x^2+y^2+z^2}} \\ z \\ 1 \end{vmatrix}$$

$$E_x = \frac{10x}{\sqrt{x^2+y^2+z^2}} + \frac{x\,z^2}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} - \frac{y}{\sqrt{x^2+y^2}}$$

$$E_y = \frac{10y}{\sqrt{x^2+y^2+z^2}} + \frac{y\,z^2}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} + \frac{x}{\sqrt{x^2+y^2}}$$

$$E_z = \frac{10z}{\sqrt{x^2+y^2+z^2}} - \frac{-z\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$E = E_x \mathrm{a}_x + E_y \mathrm{a}_y + E_z \mathrm{a}_z$$

$E\ at\ (-3\ ,4\ ,0\ )$

$$E_x = \frac{-30}{25} - \frac{4}{5} = -2$$

$$E_y = \frac{40}{25} - \frac{3}{5} = 1$$

$$E_z = 0 - 0 = 0$$

$$E = -2\mathrm{a}_x + \mathrm{a}_y$$

Homework

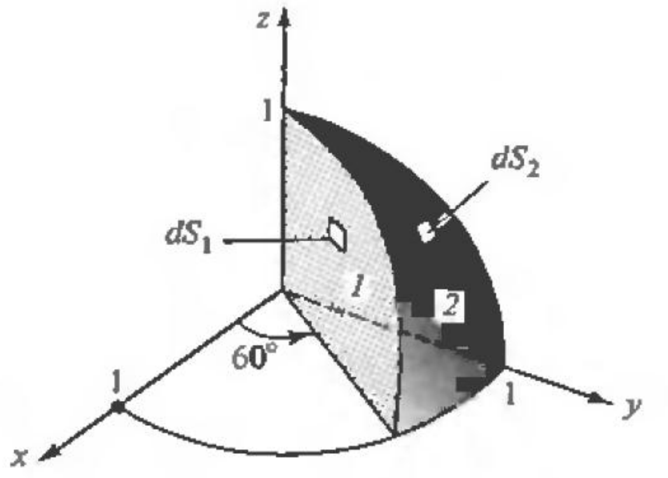
Q_1: Find the distance between $(5, 3\frac{\pi}{2}, 0)$ to $(5, \frac{\pi}{2}, 10)$ in cylindrical coordinate?

Ans: $10\sqrt{2}$

Q_2: Obtain the expression for the volume of a sphere of radius a, from the differential volume?

Q_3: Obtain the expression for the volume and surface area of a cylindrical of radius b and height h, from the differential volume?

Q_4: Using spherical coordinates to write the differential surface areas dS_1 and dS_2 and then integrate to obtain the surface marked 1 and 2 in Figure below?



Ans: $\frac{\pi}{4}, \frac{\pi}{6}$

Q_5: Express the unit vector directed toward the origin from an arbitrary point on the line described by x=0, y=3?

Ans: $\mathbf{a} = \frac{-3\mathrm{a}_y - z\mathrm{a}_z}{\sqrt{9+z^2}}$

Q_6: Find the angle between $\vec{\mathbf{A}} = 10\mathbf{a}_y + 2\mathbf{a}_z$ and $\vec{\mathbf{B}} = -4B_y\mathbf{a}_y + 0.5\mathbf{a}_z$ using both dot product and cross product?

Ans: 161.5°